On some subclasses of univalent functions

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ABSTRACT. In 1999, S. Kanas and F. Ronning introduced the classes of functions starlike and convex, wich are normalized whit $f(w)=f'(w)-1=0$ and w is a fixed point in U. In this paper we continue the investigation of the univalent functions normalized with $f(w)=f'(w)-1=0$, where w is a fixed point in U.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U=\{z\in\mathbb{C}:\ |z|<1\}$, $A=\{f\in\mathcal{H}(U):\ f(0)=f'(0)-1=0\}$ and $S=\{f\in A:\ f$ is univalent in $U\}$.

We recall here the definitions of the well - known classes of starlike, convex, close to convex and α - convex functions:

$$S^*=\left\{f\in A: Re\frac{zf'(z)}{f(z)}>0\ ,\quad z\in U\right\},$$

$$S^c=\left\{f\in A: Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0\ ,\quad z\in U\right\}$$

$$CC=\left\{f\in A: \exists\ g\in S^*,\ Re\frac{zf'(z)}{g(z)}>0\ ,\quad z\in U\right\},$$

$$M_\alpha=\left\{f\in A: \frac{f(z)\cdot f'(z)}{z}\neq 0,\ Re\ J(\alpha,f:z)>0\ ,\quad z\in U\right\}$$

where $J(\alpha,f;z)=(1-\alpha)\dfrac{zf'(z)}{f(z)}+\alpha\left(1+\dfrac{zf''(z)}{f'(z)}\right)$.

Let w be a fixed point in U and $A(w)=\{f\in\mathcal{H}(U): f(w)=f'(w)-1=0\}$.

In [3] S. Kanas and F. Ronning introduced the following classes:

$$S(w)=\{f\in A(w):\ f \text{ is univalent in } U\}$$

$$ST(w)=S^*(w)=\left\{f\in S(w): Re\frac{(z-w)f'(z)}{f(z)}>0\ ,\quad z\in U\right\}$$

$$CV(w)=S^c(w)=\left\{f\in S(w): 1+Re\frac{(z-w)f''(z)}{f'(z)}>0\ ,\quad z\in U\right\}.$$

The class $S^*(w)$ is defined by the geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$ and the corresponding class $S^c(w)$ is defined by the property that the image of any circular arc centered at w is convex. We observe that the definitions are somewhat similar to the ones for uniformly starlike and convex functions introduced by A. W. Goodman in [1] and [2], except that in this case the point w is fixed.

It is obvious that exists a natural "Alexander relation" between the classes $S^*(w)$ and $S^c(w)$:

$$g \in S^c(w) \text{ if and only if } f(z) = (z-w)g'(z) \in S^*(w).$$

Let denote with $\mathcal{P}(w)$ the class of all functions $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z-w)^n$ that are regular in U and satisfy $p(w) = 1$ and $Re\ p(z) > 0$ for $z \in U$.

2 Preliminary results

If is easy to see that a function $f_{(z)} \in A(w)$ have the series expansions:

$$f(z) = (z-w) + a_2(z-w)^2 + ...$$

In [7] J. K. Wald gives the sharp bounds for the coefficients B_n of the function $p \in \mathcal{P}(w)$:

Teorema 2.1 *If $p(z) \in \mathcal{P}(w)$, $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z-w)^n$, then*

$$|B_n| \leq \frac{2}{(1+d)(1-d)^n}, \quad \text{where } d = |w| \text{ and } n \geq 1. \tag{1}$$

Using the above result, S. Kanas and F. Ronning obtain in [3]:

Teorema 2.2 *Let $f \in S^*(w)$ and $f(z) = (z-w) + a_2(z-w)^2 + ...$ Then*

$$\begin{aligned} &|a_2| \leq \frac{2}{1-d^2}, \quad |a_3| \leq \frac{3+d}{(1-d^2)^2}, \\ &|a_4| \leq \frac{2}{3} \cdot \frac{(2+d)(3+d)}{(1-d^2)^3}, \quad |a_5| \leq \frac{1}{6} \cdot \frac{(2+d)(3+d)(3d+5)}{(1-d^2)^4} \end{aligned} \tag{2}$$

where $d = |w|$.

Remark 2.1 *It is clear that the above theorem also provides bounds for the coefficients of functions in $S^c(w)$, due to the relation between $S^c(w)$ and $S^*(w)$.*

The next theorem is the result of the so called "admissible functions method" introduced by P. T. Mocanu and S. S. Miller (see [4], [5], [6]).

Teorema 2.3 *Let h convex in U and $Re[\beta h(z)+\gamma]>0$; $z\in U$. If $p\in\mathcal{H}(U)$ with $p(0)=h(0)$ and p satisfied the Briot - Bouquet differential subordination*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec h(z)\ ,\quad \textit{then } p(z)\prec h(z).$$

3 Main results

Let consider the integral operator $L_a : A(w)\to A(w)$ defined by

$$f(z)=L_aF(z)=\frac{1+a}{(z-w)^a}\cdot\int_w^z F(t)\cdot(t-w)^{a-1}dt\ ,\quad a\in\mathbb{R}\ ,\quad a\geq 0. \tag{3}$$

We denote by $D(w)=\left\{z\in U:\ Re\left[\frac{w}{z}\right]<1\, and\, Re\left[\frac{z(1+z)}{(z-w)(1-z)}\right]>0\right\}$,with $D(0)=\bar{U}$, and $s(w)=\{f:\bar{D}(w)\to\mathbb{C}\}\cap\bar{S}(w)$, where w is a fixed point in $\bar{U}$.

Denoting $s^*(w)=\bar{S}^*(w)\cap s(w)$, where w is a fixed point in $\bar{U}$, we obtain:

Teorema 3.1 *Let w be a fixed point in U and $F(z)\in s^*(w)$. Then $f(z)=L_aF(z)\in S^*(w)$, where the integral operator L_a is defined by (3).*

Proof. By differentiating (3) we obtain:

$$(1+a)F(z)=a\cdot f(z)+(z-w)\cdot f'(z). \tag{4}$$

From (4) we have:

$$(1+a)F'(z)=(1+a)f'(z)+(z-w)f''(z). \tag{5}$$

Using (4) and (5) we obtain:

$$\frac{(z-w)F'(z)}{F(z)}=\frac{(1+a)\cdot(z-w)\cdot\dfrac{f'(z)}{f(z)}+(z-w)^2\dfrac{f''(z)}{f(z)}}{a+(z-w)\dfrac{f'(z)}{f(z)}}. \tag{6}$$

With notation $p(z)=\dfrac{(z-w)f'(z)}{f(z)}$, where $p(z)\in\mathcal{H}(U)$ and $p(0)=1$, we have:

$$(z-w)p'(z)=p(z)+(z-w)^2\cdot\frac{f''(z)}{f(z)}-[p(z)]^2$$

and thus:

(7) $$(z-w)^2\frac{f''(z)}{f(z)} = (z-w)p'(z) - p(z)[1-p(z)].$$

Using (6) and (7) we obtain:

(8) $$\frac{(z-w)F'(z)}{F(z)} = p(z) + \frac{(z-w)\cdot p'(z)}{a+p(z)}.$$

Using $F(z) \in s^*(w)$ from (8) we have:

$$p(z) + \frac{z-w}{a+p(z)} \cdot p'(z) \prec \frac{1+z}{1-z} \equiv h(z)$$

or

$$p(z) + \frac{1-\frac{w}{z}}{a+p(z)} \cdot zp'(z) \prec \frac{1+z}{1-z}.$$

From hypothesis we have $Re\left[\frac{1}{1-\frac{w}{z}} \cdot h(z) + \frac{a}{1-\frac{w}{z}}\right] > 0$ and thus from Theorem 2.3 we obtain $p(z) \prec \frac{1+z}{1-z}$ or $Re\left(\frac{(z-w)f'(z)}{f(z)}\right) > 0$, $z \in U$. This means $f(z) \in S^*(w)$.

Definition 3.1 *Let $f \in S(w)$ where w is a fixed point in U. We say that f is w - close to convex if exists a function $g \in S^*(w)$ such that $Re\frac{(z-w)f'(z)}{g(z)} > 0$, $z \in U$. We denote this class with $CC(z)$.*

Remark 3.1 *If we consider $f = g$, $g \in S^*(w)$, we have $S^*(w) \subset CC(w)$.*

If we take $w = 0$ we obtain the well known close to convex functions.

Teorema 3.2 *Let w be a fixed point in U and $f \in CC(w)$, $f(z) = (z-w) + \sum_{n=2}^{\infty} b_n \cdot (z-w)^n$, with respect to the function $g \in S^*(w)$, $g(z) = (z-w) + \sum_{n=2}^{\infty} a_n \cdot (z-w)^n$. Then*

$$|b_n| \le \frac{1}{n}\left[|a_n| + \sum_{k=1}^{n-1} |a_k| \cdot \frac{2}{(1+d)(1-d)^{n-k}}\right]$$

where $d = |w|$, $n \ge 2$ and $a_1 = 1$.

Proof. Let $f \in CC(w)$ with respect to the function $g \in S^*(w)$. Then there exists a function $p \in \mathcal{P}(w)$ such that

$$\frac{(z-w)f'(z)}{g(z)} = p(z)$$

where $p(z)=1+\sum\limits_{n=1}^{\infty}B_n(z-w)^n$.

Using the hypothesis through identification of $(z-w)^n$ coefficients we obtain:

$$n\cdot b_n=a_n+\sum_{k=1}^{n-1}a_k\cdot B_{n-k} \tag{9}$$

where $a_1=1$ and $n\geq 2$.

From (9) we have

$$|b_n|\leq\frac{1}{n}\left[|a_n|+\sum_{k=1}^{n-1}|a_k|\cdot|B_{n-k}|\right]\quad,\quad a_1=1\ ,\quad n\geq 2.$$

Applying the above and the estimates (1) we obtain the result.

Remark 3.2 *If we use the estimates (2) we obtain the same estimates for the coefficients* b_n, $n=2,3,4,5$.

Definition 3.2 *Let* $\alpha\in\mathbb{R}$ *and* w *be a fixed point in* U. *For* $f\in S(w)$ *we denote by* $J(\alpha,f,w;z)=(1-\alpha)\dfrac{(z-w)f'(z)}{f(z)}+\alpha\left[1+\dfrac{(z-w)f''(z)}{f'(z)}\right]$. *We say that* f *is* $w-\alpha$ *- convex function if* $\dfrac{f(z)\cdot f'(z)}{z-w}\neq 0$ *and* $Re\ J(\alpha,f,w;z)>0$, $z\in U$. *We denote this class with* $M_\alpha(w)$.

Remark 3.3 *It is easy to observe that* $M_\alpha(0)$ *is the well known class of* α *- convex functions.*

Teorema 3.3 *Let* w *be a fixed point in* U, $\alpha\in\mathbb{R}$, $\alpha\geq 0$ *and* $m_\alpha(w)=M_\alpha(w)\cap s(w)$.

1. *If* $f\in m_\alpha(w)$ *then* $f\in S^*(w)$. *This means* $m_\alpha(w)\subset S^*(w)$.
2. *If* $\alpha,\beta\in\mathbb{R}$, *with* $0\leq\dfrac{\beta}{\alpha}<1$, *then* $m_\alpha(w)\subset m_\beta(w)$.

Proof. From $f\in m_\alpha(w)$ we have $Re\ J(\alpha,f,w;z)>0$, $z\in U$. Using the notation $p(z)=\dfrac{(z-w)f'(z)}{f(z)}$, with $p\in\mathcal{H}(U)$ and $p(0)=1$, we obtain:

$$Re\ J(\alpha,f,w;z)=Re\left[p(z)+\alpha\cdot\frac{(z-w)p'(z)}{p(z)}\right]>0\ ,\quad z\in U\quad\text{or}$$

$$p(z)+\frac{\alpha\left(1-\frac{w}{z}\right)}{p(z)}\cdot zp'(z)\prec\frac{1+z}{1-z}\equiv h(z).$$

For $\alpha=0$ we have $p(z)\prec\dfrac{1+z}{1-z}$.

Using the hypothesis we have for $\alpha > 0$, $Re\left[\dfrac{1}{\alpha\left(1-\frac{w}{z}\right)}\cdot h(z)\right] > 0$ and from Theo= rem 2.3 we obtain $p(z) \prec \dfrac{1+z}{1-z}$.

This means that $Re\dfrac{(z=w)f'(z)}{f(z)} > 0$, $z \in U$ and $\alpha \geq 0$ or $f \in S^*(w)$.

If we denote by $A = Rep(z)$ and by $B = Re\dfrac{(z-w)p'(z)}{p(z)}$ we have $A > 0$ and $A+B\cdot\alpha >$ 0, where $\alpha \geq 0$.

Using the geometric interpretation of the equation $y(x) = A + B \cdot x$, $x \in [0,\alpha]$ we obtain

$$y(\beta) = A + B\cdot\beta > 0 \text{ for every } \beta \in [0,\alpha].$$

This means $Re\left[p(z) + \beta\cdot\dfrac{(z-w)p'(z)}{p(z)}\right] > 0$, $z \in U$ or $f \in m_\beta(w)$.

Remark 3.4 *From the above theorem we have:*

$$m_1(w) \subseteq s^c(w) \subseteq m_\alpha(w) \subseteq s^*(w)$$

where $0 \leq \alpha \leq 1$ *and* $s^c(w) = S^c(w) \cap s(w)$.

References

[1] A. W. Goodman, *On Uniformly Starlike Functions*, Journal of Math. Anal. and Appl. 155(1991), 364 - 370.

[2] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math., LVIII (1991), 86 - 92.

[3] S. Kanas and F. Ronning, *Uniformly starlike and convex functions and other related classes of univelent functions*, Annales Univ. Mariae Curie - Sklodowska, Vol. L III, 10 (1999), 95 - 105.

[4] S. S. Miller and P. T. Mocanu, *Differential subordonations and univalent functions*, Mich. Math. 28 (1981), 157 - 171.

[5] S. S. Miller and P. T. Mocanu, *Univalent solution of Briot-Bouquet differential equations*, J. Differential Equations 56 (1985), 297 - 308.

[6] S. S. Miller and P. T. Mocanu, *On some classes of first-order differential subordinations*, Mich. Math. 32(1985), 185 - 195.

[7] J. K. Wald, *On starlike functions*, Ph. D. thesis, University of Delaware, Newark, Delaware (1978).

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